Morita Equivalence of Partial Actions and Globalization.[1]

F. Abadie, M. Dokuchaev, R. Exel and J.J. Simón

Universidad de la República, Montevideo (Uruguay),
Universidade de São Paulo (Brasil)
Universidade Federal de Santa Catarina (Brasil), Universidad de Murcia (ESpaña)

[1] This work was partially supported by CNPq of Brazil (Proc. 305975/2013-7, Proc. 300362/2010-2), Fapesp of Brazil (Proc. Proc. 2009/52665-0), MINECO (Ministerio de Economía y Competitividad), (Fondo Europeo de Desarrollo Regional) project MTM2012-35240, Spain, and Fundación Séneca of Murcia, Programa Hispano Brasileño de Cooperación Universitaria PHB2012-0135, Spain.

Globalization of partial actions.

1. Since partial group actions naturally appear as restrictions of global ones, it is interesting to know when a given partial action can be obtained as a restriction of a global action.
2. Partial actions which can be obtained as restrictions of global ones are called globalizable.
3. The problem of globalization of partial actions was studied first in the F. Abadie's PhD Thesis in the context of continuous partial group actions on topological spaces and C^*-algebras.
4. In the ring theoretic context, it was studied in
   - 4.1 M. Dokuchaev, R. Exel, Associativity of crossed products by partial actions, enveloping actions and partial representations, *Trans. AMS* (2005).
   - 4.2 M. Dokuchaev, A. del Río, J. J. Simón, Globalizations of partial actions on non unital rings, *Proc. AMS*, **135** (2007).
   - 4.3 M. Dokuchaev, R. Exel, J. J. Simón, Globalization of twisted partial actions, *Trans. AMS* (2010).

Globalization of partial actions up to Morita equivalence.

Some facts about globalization

1. It is known that globalizations of partial actions on topological spaces (and abstract sets) always exist.
2. Partial actions on C^*-algebras are not globalizable in general.
3. However, F. Abadie (2003) shows that they are globalizable "up to Morita equivalence".

We want to generalize this last result in the case of abstract rings.

Globalization of partial actions up to Morita equivalence.

We proceed as follows

1. We give a concept of Morita equivalent partial actions on algebras.
2. Then we study its basic properties.
3. We show that globalization up to Morita equivalence exists.
4. We then give the notion of Morita enveloping action.
5. Then we study its properties.

Framework on partial actions

Definition

A partial action of a group G on an algebra $\mathcal{A}$, $\alpha = (\alpha_g, \mathcal{D}_g, \mathcal{A})$ satisfying the property

$$\mathcal{D}_{g_1} \cap \ldots \cap \mathcal{D}_{g_n} = \mathcal{D}_{g_1} \ldots \mathcal{D}_{g_n}, \quad \forall g_1, \ldots, g_n \in G, \ \forall n > 0$$

shall be called regular.

In particular, $\mathcal{D}_g^2 = \mathcal{D}_g$ and $\mathcal{A}\mathcal{D}_g = \mathcal{D}_g\mathcal{A} = \mathcal{D}_g$ for all $g \in G$.

An special case of regular partial actions are the s-unital partial actions.

Definition

1. A ring $\mathcal{R}$ is called right s-unital if for any $a \in \mathcal{R}$ there exists an element $x \in \mathcal{R}$ such that $ax = a$.
2. A partial action α of G on $\mathcal{A}$ is *s-unital* if $\mathcal{D}_g$ is an s-unital ring for each $g \in G$.

Morita context, context algebras and equivalences

Let $\mathcal{A}$ and $\mathcal{A}'$ be associative (not neccesarily with 1) algebras and $(\mathcal{A}, \mathcal{A}', M, M', \tau, \tau')$ a Morita context with bimodules ${}_{\mathcal{A}}M_{\mathcal{A}'}$, ${}_{\mathcal{A}'}M'_{\mathcal{A}}$, and trace maps $\tau : M \otimes_{\mathcal{A}'} M' \to \mathcal{A}$ and $\tau' : M' \otimes_{\mathcal{A}} M \to \mathcal{A}'$.

The *context ring* (or linking algebra or Morita ring) is

$$\mathcal{C} = \begin{pmatrix} \mathcal{A} & M \\ M' & \mathcal{A}' \end{pmatrix}$$

together with the obvious matrix operation.

In [J. L. García and S., *JPAA*, 1991] a Morita theory for rings without identity was developed. In particular, it was proved that the algebras $\mathcal{A}$ and $\mathcal{A}'$ are Morita equivalent if and only if the trace maps are surjective.

Morita equivalence of partial actions

Definition

Let $\alpha = (\alpha_g,\ \mathcal{D}_g,\ \mathcal{A})_G$ and $\alpha' = \left(\alpha'_g,\ \mathcal{D}'_g,\ \mathcal{A}'\right)_G$ be regular partial actions. We say that α is Morita equivalent to α' if:

1. There exists a Morita context $(\mathcal{A}, \mathcal{A}', M, M', \tau, \tau')$ with surjective trace maps.
2. There exists a product partial action $\theta = \left(\theta_g, \bar{\mathcal{D}}_{g^{-1}}, \mathcal{C}\right)_G$ where $\mathcal{C} = \begin{pmatrix} \mathcal{A} & M \\ M' & \mathcal{A}' \end{pmatrix}$, such that the restriction of θ to $\begin{pmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{pmatrix}$ is α, whereas the restriction of θ to $\begin{pmatrix} 0 & 0 \\ 0 & \mathcal{A}' \end{pmatrix}$ is α'.

Fundamental properties.

- Morita equivalence of regular partial actions is an equivalence relation.
- If $\alpha = (\alpha_g,\ \mathcal{D}_g,\ \mathcal{A})_G$ and $\alpha' = \left(\alpha'_g,\ \mathcal{D}'_g,\ \mathcal{A}'\right)_G$ are Morita equivalent regular partial actions, then the skew group rings $\mathcal{A} \rtimes_\alpha G$ and $\mathcal{A}' \rtimes_{\alpha'} G$ are Morita equivalent.

Globalization of partial actions

Definition

Let $\alpha = (\alpha_g, \mathcal{D}_g, \mathcal{A})_G$ be a partial action. A global action β of $\mathcal{B}$ is said to be a globalization of α if there exists a ring monomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ such that

1. $\varphi(\mathcal{A})$ is an ideal of B.
2. $B = \sum_{g \in G} \beta_g(\varphi(\mathcal{A}))$.
3. $\varphi(\mathcal{D}_g) = \varphi(\mathcal{A}) \cap \beta_g(\varphi(\mathcal{A}))$.
4. $\varphi \circ \alpha_g = \beta_g \circ \varphi$ over $\mathcal{D}_{g^{-1}}$.

In this case, we say that α is globalizable.

Globalization of partial actions up to Morita equivalence

Theorem

Let $\alpha = (\alpha_g, \mathcal{D}_g, \mathcal{A})_G$ be a regular partial action. Then there exists a partial action $\alpha' = (\alpha'_g, \mathcal{D}'_g, \mathcal{A}')_G$ such that

1. *α' is a regular partial action.*
2. *α' is globalizable.*
3. *α and α' are Morita equivalent partial actions.*

Construction of the globalizable partial action

Consider the skew group ring $\mathcal{A} \rtimes_\alpha G$, set $\mathcal{B}_g = \mathcal{D}_g \delta_g$, and construct the G-graded ring $\mathcal{B} = \oplus_{g \in G} \mathcal{B}_g$.

We set

- $\mathcal{E} = \mathrm{FM}_{g,h \in G}\left(\mathcal{B}_{g^{-1}h}\right) \subseteq \mathrm{FM}_G(\mathcal{B})$, $\quad (a_{(g,h)} \in \mathcal{B}_{g^{-1}h})$
- $\mathcal{A}' = \mathrm{FM}_{g,h \in G}\left(\mathcal{B}_{g^{-1}}\mathcal{B}_h\right) \subseteq \mathcal{E}$, $\quad (a_{(g,h)} \in \mathcal{B}_{g^{-1}}\mathcal{B}_h)$
- $M = \oplus_{h \in G} \mathcal{B}_1 \mathcal{B}_h$ $\quad (\mathcal{B}_1, \mathcal{A}')$-bimodule.
- $M' = \oplus_{h \in G} \mathcal{B}_{g^{-1}} \mathcal{B}_1$ $\quad (\mathcal{A}', \mathcal{B}_1)$-bimodule.

and construct the Morita context between rings

$$(\mathcal{B}_1, \mathcal{A}', M, M', \tau, \tau')$$

where the trace maps are obvious multiplication.

Facts:

- Trace maps are onto so that the algebras are Morita equivalent.
- $\mathcal{A} \cong \mathcal{B}_1$ as algebras.

Construction of the globalizable partial action

We define the (global) action β of G on $\mathcal{E}$ via permutation of rows and columns as

$$\beta_f(a)_{(g,h)} = a_{(f^{-1}g, f^{-1}h)}$$

and then we restrict it obtaining the partial action

$$\alpha'_g = \beta_{g_|} : \mathcal{D}'_{g^{-1}} \to \mathcal{D}'_g \text{ where } \mathcal{D}'_h = \mathcal{A}' \cap \beta_h(\mathcal{A}').$$

One may prove that the equivalence $(\mathcal{A}, \mathcal{A}', M, M', \tau, \tau')$ satisfies

$$M'\mathcal{D}_g M = \mathcal{D}'_g \quad \text{and} \quad M\mathcal{D}'_g M' = \mathcal{D}_g$$

and hence

The partial actions $\alpha = (\alpha_g, \mathcal{D}_g, \mathcal{A})_G$ and $\alpha' = (\alpha'_g, \mathcal{D}'_g, \mathcal{A}')_G$ are Morita equivalent.

Uniqueness of the globalization

Definition

Let $\alpha = (\alpha_g, \mathcal{D}_g, \mathcal{A})_G$ be a partial action. An enveloping action of α is a (global) action β of G on $\mathcal{B}$ such that:

1. $\mathcal{A}$ is an ideal of $\mathcal{B}$ and α is the restriction of β to $\mathcal{A}$.
2. $\mathcal{B} = \sum_{g \in G} \beta_g(\mathcal{A})$

If α' is a regular partial action Morita equivalent to α, and β' is an enveloping action of α', we say that β' is a Morita enveloping action of α.

It is proved that

Theorem

If $\alpha = (\alpha_g, \mathcal{D}_g, \mathcal{A})_G$ is a regular partial action then α has a Morita enveloping action, which is unique up to Morita equivalence of partial actions.
Moreover, all skew group rings of the globalizations are Morita equivalent rings.

Stable equivalences

Definition

Let $\mathcal{A}$ be a ring without identity. We say that $\mathcal{A}$ has orthogonal local units (or that $\mathcal{A}$ has enough idempotents) if there is a set $E \subset \mathcal{A}$, such that

1. E is a set of pairwise orthogonal idempotents.
2. $\mathcal{A} = \oplus_{e \in E} \mathcal{A}e = \oplus_{e \in E} e\mathcal{A}$.

Remarks

1. *Every Morita equivalent class of rings with local units has at least one representative with orthogonal local units (Ánh and Marki, Tsukuba J. Math. 1987).*
2. *Is $\mathcal{A}$ has a countable set of local units then $\mathcal{A}$ has a set of orthogonal local units (P. Ara, Algebra Colloq. 2004).*

Stable equivalences. Setting the problem.

Theorem (Dokuchaev, Exel, Simón, 2008)

Let $\mathcal{A}$ and $\mathcal{A}'$ be algebras with orthogonal local units (OLU, for short) which are Morita equivalent. Then there exists a cardinal number γ such that $\mathrm{FM}_\gamma(\mathcal{A}) \cong \mathrm{FM}_\gamma(\mathcal{A}')$.

Setting the problem

Let $\mathcal{A}$ and $\mathcal{A}'$ be algebras with OLU and let $\alpha = (\alpha_g, \mathcal{D}_g, \mathcal{A})$ and $\alpha' = \left(\alpha'_g, \mathcal{D}'_g, \mathcal{A}'\right)$ Morita equivalent partial actions. Then

- Their partial skew group rings $\mathcal{A} \rtimes_\alpha G$ and $\mathcal{A}' \rtimes_{\alpha'} G$ have OLU, and hence
- There exists a cardinal number γ such that $\mathrm{FM}_\gamma(\mathcal{A} \rtimes_\alpha G) \cong \mathrm{FM}_\gamma(\mathcal{A}' \rtimes_{\alpha'} G)$
- $\mathrm{FM}_\gamma(A \rtimes_\alpha G)$ may be identified with $\mathrm{FM}_\gamma(A) \rtimes_\theta G$ (where θ is the obvious partial action in matrices) and hence it is a graded ring.

Stable equivalences. Setting the problem.

First question

Is the isomorphism

$$\mathrm{FM}_\gamma(\mathcal{A}) \rtimes_\theta G \cong \mathrm{FM}_\gamma(\mathcal{A}') \rtimes_{\theta'} G$$

a graded isomorphism?

YES

Second question

Are the partial actions isomorphic?STOP.

What is an isomorphism of partial actions?

Isomorphisms of partial actions

Definition

Two partial actions $\alpha = (\alpha_g, \mathcal{D}_g, \mathcal{A})$ and $\alpha' = \left(\alpha'_g, \mathcal{D}'_g, \mathcal{A}'\right)$ are isomorphic if there exists an isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ such that

1. $\varphi(\mathcal{D}_g) = \mathcal{D}'_g$ $(\forall\, g \in G)$.
2. $\varphi(\alpha_g(a)) = \alpha'_g(\varphi(a))$, for all $a \in \mathcal{D}_{g^{-1}}$

We answer the second question

Let $\alpha = (\alpha_g, \mathcal{D}_g, \mathcal{A})$ and $\alpha' = \left(\alpha'_g, \mathcal{D}'_g, \mathcal{A}'\right)$ be partial actions. Suppose α and α' are s-unital. If the skew group rings $\mathcal{A} \rtimes_\alpha G$ and $\mathcal{A}' \rtimes_{\alpha'} G$ are isomorphic, as graded rings, then α and α' are isomorphic partial actions.

Application to algebras with orthogonal local units

So, suppose that $\mathcal{A}$ and $\mathcal{A}'$ are algebras with OLU and let $\alpha = (\alpha_g, \mathcal{D}_g, \mathcal{A})$ and $\alpha' = \left(\alpha'_g, \mathcal{D}'_g, \mathcal{A}'\right)$ be Morita equivalent partial actions. Then

1. There exists a cardinal number γ such that $\mathrm{FM}_\gamma(A) \rtimes_\theta G \cong \mathrm{FM}_\gamma(A') \rtimes_{\theta'} G$, where θ is the partial action of matrices mentioned above. Hence
2. The partial actions θ and θ' are isomorphic

Isomorphisms. Commutative rings

As in the case of Morita theory for rings with 1, we have:

Known fact:

If $\mathcal{A}$ and $\mathcal{A}'$ are Morita equivalent commutative idempotent rings then they are isomorphic rings (García and S., *JPAA* 1991).

In the case of partial actions we have

Theorem

Let $\alpha = (\alpha_g, \mathcal{D}_g, \mathcal{A})$ and $\alpha' = (\alpha'_g, \mathcal{D}'_g, \mathcal{A}')$ be Morita equivalent, s-unital partial actions. If $\mathcal{A}$ and $\mathcal{A}'$ are commutative rings then α and α' are isomorphic partial actions.

And,

Theorem

Let $\mathcal{A}$ and $\mathcal{A}'$ be commutative C^*-algebras and $\alpha = (\alpha_g, \mathcal{D}_g, \mathcal{A})$ and $\alpha' = (\alpha'_g, \mathcal{D}'_g, \mathcal{A}')$ be Morita equivalent partial actions. Then α and α' are isomorphic partial actions.

Thank you